# 1 Three dimensional mirror symmetry

## 1.1 3d $\mathcal{N} = 4$ SCFT

The main physical theories we are interested in are three dimensional $\mathcal{N} = 4$ superconformal field theory (SCFT). There are some basic facts about these theories

- The bosonic symmetry group is $SO(2,3) \times SU(2)_1 \times SU(2)_2 \times G$, here $SO(2,3)$ is the conformal group and $SU(2)_1$ and $SU(2)_2$ are the R symmetry groups. $G$ is other global symmetry group, which might be absent.

- These theories can have moduli space of vacua, which often has two components

  1. One component can be called Coulomb branch $M_C$ (This name is related to the gauge theory origin which we will discuss later), and $SU(2)_1$ acts on it while $SU(2)_2$ acts trivially. There might be some global symmetry group $G_1$ acting on it.
  2. One component can be called Higgs branch $M_H$, and $SU(2)_2$ acts on it while $SU(2)_1$ acts non-trivially. There might be some global symmetry group $G_2$ acting on it.

  Each component is therefore a cone and the low energy theory on it can be described by a non-linear sigma model, where the target is $M_C$ or $M_H$ with a hyperkhaler metric. (There are actually more general low energy behaviors which we could discuss later too). Some interesting physical quantities we would like to compute are:

  - The global symmetry group $G_1$ and $G_2$.
  - The hyperkhaler metric on each component $M_C$ and $M_H$, which is probably too difficult. It is less difficult to determine the holomorphic structure. Physically, the moduli space on each components are parameterized by the expectation value of half-BPS operators. These operators form a ring which might be called **chiral ring**, which is identified with the coordinate ring of the moduli space.
  - Some other observables, such as the supersymmetric index, partition function on various curved manifolds such as $S^3$, etc.

## 1.2 Constructing SCFT

So how can we construct 3d $\mathcal{N} = 4$ SCFT? There are several interesting ways:

- They are found as the IR fixed point of 3d $\mathcal{N} = 4$ gauge theories. 3d gauge interactions are relevant, so they typically give an interacting fixed point in the IR limit. A large class of such examples are given by the so-called quiver gauge theories. Now for a general quiver gauge theory, the Higgs branch usually is not changed under RG flow, and it can be computed classically, which is actually the same as the Nakajima quiver variety. Generally the Coulomb branch chiral ring is not changed under RG flow (the hyperkhaler metric does change!), and one could use the gauge theory description to describe it. The global symmetry on moduli space can be computed too, and the main complication is the symmetry on Coulomb branch, and there is a simple way to study them (Gaiotto-Witten).

- We now start with four dimensional $\mathcal{N}=2$ SCFT, and put it on a circle $S^1$, and then flow to the IR. The interesting point is that the space of 4d $\mathcal{N}=2$ SCFT is increased greatly in recent years, and so correspondingly this construction would give a large class of new 3d $\mathcal{N}=4$ SCFT. Certain class is studied by (Benini-Tachikwa-Xie), which leads to Star-shaped quiver. Much more general classes (called Argyres-Douglas SCFT in 4d, see Xie) have not been studied yet. One class is related to fusion quiver studied by Boalch.

- String theory construction. The above two class of constructions can be embedded in string theory. One type construction is the brane construction by Hanany-Witten, and the other type is the so-called 6d $(2,0)$ theory (Gaiotto, & Xie), and string theory on three-fold singularity (Xie, Yau).

## 1.3 Mathematics involved

Some mathematical connections:

- Since the moduli space of 3d $\mathcal{N}=4$ SCFT is a hyperkahler manifold, so hyperkhaler manifold is naturally a very interesting subject. The basic facts of hyperkhaler manifold and the relation with supersymmetry is explained in (Hitchim, Rocek, et al). These hyperkhaler manifolds appear in the following context: instanton moduli space (Kronheim-Nakajima), moduli space of Nahm equation with specified boundary conditions (Kronheim), associated variety of vertex operator algebra (Arakawa).

- Representation theory. Many moduli space is related to co-adjoint orbit of Lie algebra. Nilpotent orbits, and the generalized Slowdoy slices appear naturally. The associated variety of primitive ideal also appears in the VOA context.

- Resolution of singularity. The moduli space is a hyperkahler cone, and if there are global symmetry acting on it, one can turn on deformation which will deform the singularity.

- Rozansky-Witten invariant: Rozansky-Witten constructed a TQFT starting from a 3d $\mathcal{N}=4$ SCFT.

- Mathematical construction of Coulomb branch (Nakajima, et al)!

- Any mathematical questions on hyperkahler cone should have a physical interpretation for 3d $\mathcal{N}=4$ SCFT. For instance, finite W algebra, which might be related to line defects of field theory.

- Geometric Langlands: some of 3d $\mathcal{N} = 4$ SCFTs play important role in the study of geometric Langlands program.
- Hitchin system: some moduli space would appear as certain limit of Hitchin moduli space.

### 1.4 3d mirror symmetry

Intriligator and Seiberg found following interesting duality between two 3d $\mathcal{N} = 4$ SCFT:

- There are two SCFT $T_A$ and $T_B$ which is dual to each other. The basic map is: The Higgs branch of $T_A$ is mapped to the Coulomb branch of $T_B$, and vice versa.

This duality is quite useful in one aspect: if our theory has a quiver gauge theory description, then the Higgs branch is easy to describe while the Coulomb branch is hard. So using mirror symmetry, we can map the hard computation of Coulomb branch of one theory to the computation of the Higgs branch of the other theory.

### 1.5 Constructing mirror pair

There are several ways to give the mirror pairs:

- Brane construction: Hanany-Witten shows that one can interpret the Mirror symmetry as the S duality of type IIB string theory. Their constructions are mostly applied to the linear quiver case. Later on, Gaiotto-Witten used brane construction to construct the mirror pairs, whose moduli space of vacua is given by nilpotent orbit and generalized slowdoy slice.
- Class S construction: Benini-Tachikawa-Xie constructed mirror pairs using the $(2, 0)$ construction. One side is the star-shaped quiver. Xie constructed mirror pair involves the fusion quiver of Boalch, and the mirror is the 4d Argyres-Douglas theory compactified to circle. This class of theories are much less studied.
- Associated variety: Xie-Yan computed the associated variety of a large class of 4d N=2 SCFT, therefore gives many interesting new mirror pairs (the mirror is not clear though)!
- Mirror symmetry of 3-fold singularity. Here the Coulomb branch and Higgs branch is related to resolution and deformation of the singularity. This approach is much less studied.
- If the gauge groups are $U(1)$, so the moduli space is toric, one have a way to find its mirror.

**Example**: Let's consider following quiver

$$U(1) - U(2) - \ldots - U(N-1) - [U(N)] \tag{1}$$

This theory flows to a 3d $\mathcal{N}=4$ SCFT in the IR, and the global symmetry of the IR theory is $SU(N) \times SU(N)$. This theory is self-mirror, and the Higgs branch is isomorphic to Coulomb branch, which is just $T^*(G/B)$, with $G = SU(N)$ and $B$ is the Borel subgroup. This example can be generalized by arbitrary $G$, and $M_H$ and $M_V$ are $T^*(G/B)$ and $T^*(G^\vee/B^\vee)$. More generally, one could find theory whose moduli space is given by $T^*(G/P)$ with $P$ a general parabolic group.

# 2 Symplectic duality

So one can attach two interesting moduli spaces $M_H$ and $M_V$ to a single SCFT. There would be physical observables of SCFT which can be computed both from $M_H$ and $M_V$. The objects appearing in two different computations would be quite different. Here are some examples

- To compute the partition function of SCFT, one could use the Higgs branch localization or Coulomb branch localization (Benini,et al), and the results are the same, but the computations involve very different objects.
- One can consider Higgs branch $M_H(m)$ and also include the mass deformation (mathematically turning on mass deformation would deform the singularity), and one can associate a category $\mathcal{O}_1$, which is related to the quantization of $M_H(m)$. On the other hand, one can consider Coulomb branch $M_C(\zeta)$ and the FI deformation (mathematically turning on FI parameter would resolve the singularity), and one can associate another category $\mathcal{O}_2$, which is again related to the quantization of $M_C(\zeta)$. Symplectic duality claims that these two categories are equivalent with some non-trivial transformations.
- In case the moduli space is the cotangent bundle of flag variety $T^*(G/P)$, it seems that the categories constructed has representation meaning, so this fits in the study of geometric representation theory.

# 3 The goal

New examples of symplectic duality pairs motivated from physics. Better understanding of 3d mirror symmetry and symplectic duality. Relation to geometric Langlands, deformation quantization, brane quantization, fukaya catetories, VOA, etc.