Overview

1) The lecture will focus on “topological quantum field theories” (TQFTs) in 1+1d (chiral conformal field theory, conformal blocks) and 2+1d (CS-theory, Jones polynomial).

2) A TQFT is a quantum field theory which computes “topological invariants” → correlation/partition functions do not depend on metric of spacetime
   (In a chiral CFT they do depend on the complex structure, so the above notion has to be relaxed somewhat.)

→ TQFTs are applied to curved spacetimes, such as Riemann surfaces $\Sigma$ and 3-manifolds $M$. 
3) Applications: Non-abelian anyons

- quantum statistics in 3+1 D:
  interchanging two particles twice gives

\[ \Psi(\vec{r}_1, \vec{r}_2) \rightarrow \pm \Psi(\vec{r}_1, \vec{r}_2) \]

\[ \uparrow : \text{bosons} \]
\[ \downarrow : \text{fermions} \]

- quantum statistics in 2+1 D:

\[ \Psi(\vec{r}_1, \vec{r}_2) \rightarrow e^{i\theta} \Psi(\vec{r}_1, \vec{r}_2) \quad (a) \Leftrightarrow (b) \]

\[ \Rightarrow \text{"anyons"} \]

general case of N particles:

\[ \Rightarrow \text{"braid group" } B_N \]
generators of braid group:

\[ \sigma_i \rightarrow \sigma_{i+1} = \sigma_{i+1} \sigma_i \sigma_{i+1} \]

for 3 particles. In general \( \sigma_1, \ldots, \sigma_{N-1} \)

braid relation:

\[ \text{Important relations in a TQFT:} \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \]

also: \( \sigma_i \sigma_j = \sigma_j \sigma_i \) for \( |i-j| \geq 2 \)

\( \sigma_i^2 \neq 1 \Rightarrow \) infinitely many elements

- abelian representations of \( \mathbb{B}_N \):

\[ \Psi(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N) \mapsto e^{im \theta} \Psi(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N) \]

where \( m = \# \begin{array}{c} \sigma_i \end{array} - \# \begin{array}{c} \sigma_i \end{array} \)

for identical particles. Non-identical case:

\[ \Theta_{ab}, \quad a, b = 1, \ldots, n_s \]

number of particle species

- non-abelian repr. of \( \mathbb{B}_N \):

degenerate states

\[ \Psi x, \quad x = 1, 2, \ldots, g \]

Then \( \sigma_i \) act as \( g \times g \) unitary matrix \( \rho(\sigma_i) \)
\( \Psi_2 \rightarrow [ \rho(\sigma_1)]_{\Psi_2} \Psi_0 \)

in particular: \( \rho(\sigma_1)\rho(\sigma_2) \neq \rho(\sigma_2)\rho(\sigma_1) \)

\( \Rightarrow \) “non-abelian braiding statistics”

\underline{Example}:

Consider a model with 3 anyon types:

1, \( \sigma \), \( \Psi \)

with “fusion rules”:

\( \sigma \times \sigma = 1 + \Psi \), \( \sigma \times \Psi = \Psi \), \( \Psi \times \Psi = 1 \),

\( 1 \times x = x \) for \( x = 1, \sigma, \Psi \)

\( \rightarrow \bullet \leftarrow \)

\( \Rightarrow \bullet \)

a)

fusion channels

\( \Rightarrow \bullet \)

b)

Note the similarity to tensor products of \( SU(2) \) representations:

\( \frac{1}{2} \times \frac{1}{2} = 0 + 1 \), \( \frac{1}{2} \times 1 = \frac{1}{2} + \frac{1}{2} \), \( 1 \times 1 = 0 \)

\( \sigma \times \sigma = \Psi \), \( \sigma \times \Psi = \Psi \), \( \Psi \times \Psi = 1 \)

with important constraint: maximum spin = 1

\( \rightarrow \) Will see how these fusion rules arise in a TQFT later on.
Hilbert space of 4 T five group according to (1,2) and (3,4)

\[ \psi \]

constraint: global topological charge \( Q \) = 1

\[ \psi_1 \text{ and } \psi_2 \text{ fuse to } 1 \left( \psi_3 \text{ and } \psi_4 \text{ too} \right) \]

or \( \psi_1 \text{ and } \psi_2 \text{ fuse to } \psi \left( \psi_3 \text{ and } \psi_4 \text{ too} \right) \)

\[ \Rightarrow \text{ two-dim Hilbert space generated by } \psi_1 \text{ and } \psi_2 \]

In general: for \( 2n \) quasi-particles

Hilbert space is \( 2^{n-1} \) dimensional.

Action of braid group generators:

spinor representation of \( SO(2n) \)

braiding particles \( i \) and \( j \)

\[ \Rightarrow \pi \text{ rotation in the } i-j \text{ plane of } \mathbb{R}^{2n} \]

Realization in TQFT:

We will see in the course of these lectures that wavefunctions can equivalently be described in terms of correlation functions of TQFTs:

\[ \mathcal{H}(\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_N) = \langle \mathcal{Q} \mathcal{Q}_{N1} \rangle_{\text{TQFT}} \]
4) Abstract definition of 2+1d TQFTs

A TQFT in 2+1 dimensions (more general D+1 dimensions) is a functor \( Z \) satisfying the following conditions:

1. For each compact oriented Riemann surface \( \Sigma \) without boundary, \( \rightarrow \) complex vector space \( Z_{\Sigma} \).

2. A compact oriented 3-dimensional smooth manifold \( Y \) with \( \partial Y = \Sigma \) determines a vector \( Z(Y) \in Z_{\Sigma} \).

Furthermore, \( Z \) satisfies the following axioms:

(A1) Let \( -\Sigma \) be \( \Sigma \) with reverse orientation, \( \rightarrow Z_{-\Sigma} = Z_{\Sigma}^* \) (dual vector space).

(A2) \( Z_{\Sigma_1 \cup \Sigma_2} = Z_{\Sigma_1} \otimes Z_{\Sigma_2} \)

A 3-manifold \( Y \) with \( \partial Y = (-\Sigma_1) \cup \Sigma_2 \) \( \rightarrow \) linear map \( Z(Y) \in \text{Hom}(Z_{\Sigma_1}, Z_{\Sigma_2}) \)

\[ Z_{\Sigma_1} \xrightarrow{Z(Y)} Z_{\Sigma_2} \]

"cobordism"
(A3) \( \exists Y_1 = (- \Sigma_1) \cup (\Sigma_2) \) and \( \exists Y_2 = (- \Sigma_2) \cup \Sigma_3 \)
\[ \Rightarrow \mathcal{Z}(Y_1 \cup Y_2) = \mathcal{Z}(Y_2) \circ \mathcal{Z}(Y_1) \]

(A4) For an empty set \( \emptyset \) we have \( \mathcal{Z}(\emptyset) = \mathbb{C} \)

(A5) Let \( I \) denote the closed unit interval. Then, \( \mathcal{Z}(\Sigma \times I) \) is the identity map as a linear transformation of \( \mathbb{Z}_\Sigma \).

Lecture Content

1) Conformal Field Theory & Topology
   - Loop groups and affine Lie algebras
   - Representations of affine Lie algebras
   - Wess-Zumino Witten model
   - The space of conformal blocks
   - KZ equation
   - Vertex operators and OPE
2) Chern-Simons Theory
   • KZ equations and representations of braid groups
   • Conformal field theory and the Jones polynomial
   • Witten's invariants for 3-manifolds
   • Projective representations of mapping class groups
   • Chern-Simons theory and connections on surfaces

3) Applications of CS-theory: Non-Abelian anyons
   • Non-Abelian braiding statistics
   • Emergent Anyons
   • Review of Quantum Hall Physics
   • Quantum Hall wave-functions from conformal field theory