Symmetry

Quantum mechanics

- states. $|\psi\rangle \in \mathcal{H}$ $\quad \langle\psi|$ $\quad |\langle\psi|\psi\rangle| = 1$
- probability: $P(|\psi\rangle \to |\Phi\rangle) = |\langle\Phi|\Psi\rangle|^2$
- symmetry transformation: $\mathcal{U}: \mathcal{H} \longrightarrow \mathcal{H}$

$$|\psi\rangle \longrightarrow |\psi'\rangle$$

s.t. $P(|\psi\rangle \to |\Phi\rangle) = P(|\psi'\rangle \to |\Phi'\rangle)$

Theorem (Wigner 1930s)

Any such $\mathcal{U}$ has to be unitary and linear

$$\langle \mathcal{U}\Phi | \mathcal{U}\psi\rangle = \langle \Phi|\psi\rangle$$

$$\mathcal{U}(\xi|\psi\rangle + \eta|\Phi\rangle) = \xi|\mathcal{U}\psi\rangle + \eta|\mathcal{U}\Phi\rangle$$

or antiunitary and anti-linear

$$\langle \mathcal{U}\Phi | \mathcal{U}\psi\rangle = \langle \psi|\Phi\rangle^*$$

$$\mathcal{U}(\xi|\psi\rangle + \eta|\Phi\rangle) = \xi^*|\mathcal{U}\psi\rangle + \eta^*|\mathcal{U}\Phi\rangle$$

- adjoint of $\mathcal{U}$: $\langle \mathcal{U}\Psi|\Phi\rangle = \langle\psi|\mathcal{U}^\dagger\Phi\rangle$
- unitary: $\mathcal{U}^\dagger = \mathcal{U}^{-1}$

Symmetry groups G

- identity transformation: 1 always a symmetry
- composition of transformation: product
- inverse of transformation

$\mathcal{H}$: a unitary representation of G

$$g \longrightarrow U(g) \qquad U(g_2)U(g_1) = e^{i\phi(g_2,g_1)} U(g_2 g_1)$$

projective rep.

Groups : finite , Lie

- Lie group : $g(\theta)$ $\quad \theta^a$: continuous parameters

$$\theta \ll 1 \quad : U(g(\theta)) = 1 + \theta^a T_a + \cdots$$

anti-Hermitian

$$g(\theta_1) g(\theta_2) g^{-1}(\theta_1) g^{-1}(\theta_2)$$

$$\longrightarrow [T_a, T_b] = f_{ab}{}^{c} T_c \qquad \text{Lie algebra}$$

- Abelian $\quad f_{ab}{}^{c} = 0$
- def $\Rightarrow f_{ab}{}^{c} = - f_{ba}{}^{c}$

Symmetry of classical fields

$$S[\phi(x)] = \int d^d x \, \mathcal{L}(\phi^a(x))$$

- symmetry transformation (infinitesimal version)

$$\phi^a(x) \longrightarrow \phi^a(x) + \varepsilon \mathcal{F}^a(\phi, \partial\phi), \quad \text{s.t } \delta S = 0$$

global : ε is a constant

local : ε is space-time dependent.

note: as we will see:

- global symmetry maps one physical state to another physical state
- local symmetry maps different description of the same physical state (redundancy)
- duality: theories with different local symmetry can be equivalent, but their global symmetry has to be the same

Noether's theorem: symmetries imply conservation laws

sketch of the proof.

$$0 = \delta S = \varepsilon \int d^d x \, \frac{\delta S[\phi]}{\delta \phi^a(x)} \mathcal{F}^a(x)$$

$$\text{under } \phi^a(x) \longrightarrow \phi^a(x) + \varepsilon \mathcal{F}^a(x)$$

promote ε to $\varepsilon(x)$

$$\delta S = -\int d^d x \, J^\mu(x) \frac{\partial \varepsilon(x)}{\partial x^\mu}, \quad \text{not zero in general}$$

now consider $\phi^a(x)$ satisfy classical equation of motion
(stationary point of S, i.e. $\delta S = 0$ for any $\varepsilon(x)$)

$$\Rightarrow \delta S = -\int d^d x \, J^\mu(x) \frac{\partial \varepsilon(x)}{\partial x^\mu} = 0$$

when $\phi^a(x)$ satisfy e.o.m

integrating by parts: $\partial_\mu J^\mu(x) = 0$
$\downarrow$
conservation law.

note: easiest way to get $J^\mu(x)$ for a global symmetry:
promote it to a local symmetry, then find the term proportional to $\partial_\mu \varepsilon(x)$, the coefficient is $J^\mu(x)$

Ex: massless complex scalar in 3+1 d

$$\mathcal{L} = -\partial^\mu \phi^* \partial_\mu \phi$$

$u(1)$ global symmetry, $\phi \to e^{i\varepsilon} \phi$

local $\quad \phi \to e^{i\varepsilon(x)} \phi$

$$\delta \mathcal{L} = (i \phi^* \partial_\mu \phi) \partial^\mu \varepsilon(x) - i (\partial^\mu \phi^*) \phi \partial_\mu \varepsilon(x)$$

$$= i(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*) \partial^\mu \varepsilon(x)$$

$u(1)$ current: $\quad J_\mu = i(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*)$

$u(1)$ charge: $\quad Q = \int d^3 x \, J^0$